MARKOV ELEMENTS IN AFFINE TEMPERLEY-LIEB ALGEBRAS

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Abstract. We define a tower of affine Temperley-Lieb algebras of type $\tilde{A}_n$ and we define Markov elements in those algebras. We prove that any trace over an affine Temperley-Lieb algebras of type $\tilde{A}_2$ is uniquely defined by its values on the Markov elements.

1. Introduction

In [5] we define a tower $(\widehat{TL}_{n+1}(q))_{n\geq 0}$ of affine Temperley-Lieb algebras of type $\tilde{A}_n$ and we prove that there exists a unique Markov trace on this tower. Crucial in the proof is the definition of *Markov elements* and the following Theorem :

Theorem 1.1. *Any trace over $\widehat{TL}_{n+1}(q)$ for $2 \leq n$ is uniquely defined by its values on the Markov elements in $\widehat{TL}_{n+1}(q)$.*

The proof of this Theorem for $3 \leq n$ is given in [3], where we have omitted the case $n = 2$, long and technical. We thus present it here for completeness.

2. Notations

Let K be an integral domain of characteristic 0. Suppose that q is a square invertible element in K of which we fix a root $\sqrt{q}$. For x, y in a given ring we define $V(x, y) := xyx + xy + yx + x + y + 1$. We mean by algebra in what follows K-algebra.

We denote by $B(\tilde{A}_n)$ (resp. $W(\tilde{A}_n)$) the affine braid (resp. affine Coxeter) group with $n + 1$ generators of type $\tilde{A}$, while we denote by $B(A_n)$ (resp. $W(A_n)$) the braid (resp. Coxeter) group with n generators of type A, where $n \geq 0$. Let $W^c(\tilde{A}_n)$ (resp. $W^c(A_n)$) be the set of fully commutative elements in $W(\tilde{A}_n)$ (resp. $W(A_n)$).

Let $n \geq 2$. We define $\widehat{TL}_{n+1}(q)$ to be the algebra with unit given by a set of generators $\left\{g_{\sigma_1}, ..., \; g_{\sigma_n}, g_{a_{n+1}}\right\}$, with the following relations [1]:

- $g_{\sigma_i} g_{\sigma_j} = g_{\sigma_j} g_{\sigma_i}$, for $1 \leq i, j \leq n$ and $|i - j| \geq 2$.
- $g_{\sigma_i} g_{a_{n+1}} = g_{a_{n+1}} g_{\sigma_i}$, for $2 \leq i \leq n - 1$.
- $g_{\sigma_i} g_{\sigma_{i+1}} g_{\sigma_i} = g_{\sigma_{i+1}} g_{\sigma_i} g_{\sigma_{i+1}}$, for $1 \leq i \leq n - 1$.
- $g_{\sigma_i} g_{a_{n+1}} g_{\sigma_i} = g_{a_{n+1}} g_{\sigma_i} g_{a_{n+1}}$, for $i = 1, n$.

- $g_{\sigma_i}^2 = (q-1)g_{\sigma_i} + q,$ for $1 \leq i \leq n.$
- $g_{a_{n+1}}^2 = (q-1)g_{a_{n+1}} + q,$
- $V(g_{\sigma_i}, g_{\sigma_{i+1}}) = V(g_{\sigma_1}, g_{a_{n+1}}) = V(g_{\sigma_n}, g_{a_{n+1}}) = 0,$ for $1 \leq i \leq n-1.$

The set $\left\{g_w : w \in W^c(\tilde{A}_n)\right\}$ is well defined in the usual sense of the theory of Hecke algebra and it is a K-basis. We set $T_{a_{n+1}}$ (resp. T_{σ_i} for $1 \leq i \leq n$) to be $\sqrt{q}g_{a_{n+1}}$ (resp. $\sqrt{q}g_{\sigma_i}$ for $1 \leq i \leq n$). Hence, T_w is well defined for $w \in W^c(\tilde{A}_n)$, it equals $q^{\frac{l(w)}{2}}g_w$. The multiplication associated to the basis $\left\{T_w : w \in W^c(\tilde{A}_n)\right\}$, is given as follows:

$$\begin{aligned} &T_w T_v = T_{wv} && \text{whenever } l(wv) = l(w) + l(v). \\ &T_s T_w = \sqrt{q}(q-1)T_w + q^2 T_{sw} && \text{whenever } l(sw) = l(w) - 1, \end{aligned}$$

for w, v in $W^c(\tilde{A}_n)$ and s in $\{\sigma_1, ..., \ \sigma_n, a_{n+1}\}$.

In what follows we suppose that $q+1$ is invertible in K, we set $\delta = \frac{1}{2+q+q^{-1}} = \frac{q}{(1+q)^2}$ in K. In view of [2] , for $1 \leq i \leq n$ we set $f_{\sigma_i} := \frac{g_{\sigma_i}+1}{q+1}$ and $f_{a_{n+1}} := \frac{g_{a_{n+1}}+1}{q+1}$. In other terms $g_{\sigma_i} = (q+1)f_{\sigma_i} - 1$, and $g_{a_{n+1}} = (q+1)f_{a_{n+1}} - 1$. The set $\left\{f_w : w \in W^c(\tilde{A}_n)\right\}$ is well defined and it is a K-basis for $\widehat{TL}_{n+1}(q)$.

We define the Temperley-Lieb algebra of type A with n generators $TL_n(q)$, as the subalgebra of $\widehat{TL}_{n+1}(q)$ generated by $\{g_{\sigma_1}, ..., \ g_{\sigma_n}\}$, with $\{g_w : w \in W^c(A_n)\}$ as K-basis.

Now for $TL_0(q) = K$, we consider the following tower:

$$TL_0(q) \ \subset TL_1(q) \ \ ... \subset TL_{n-1}(q) \quad \subset TL_n(q) \ \ ...$$

Theorem 2.1. *[6] There is a unique collection of traces $(\tau_{n+1})_{0\leq n}$ on $(TL_n)_{0\leq n}$, such that:*

(1) $\tau_1(1) = 1$.

(2) *For $1 \leq n$, we have $\tau_{n+1}(hT_{\sigma_n}^{\pm 1}) = \tau_n(h)$, for any h in $TL_{n-1}(q)$.*

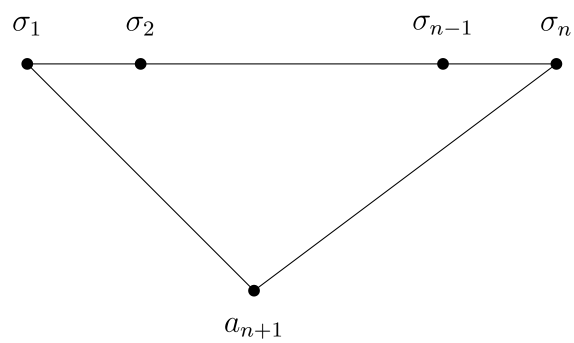
The collection $(\tau_{n+1})_{0\leq n}$ is called a Markov trace. Moreover, for any a, b and c in $TL_n(q)$ and for $n \geq 1$, every $\tau_{n+1} : TL_n(q) \longrightarrow K$ verifies:

$$\tau_{n+1}(bT_{\sigma_n}c) = \tau_n(bc) \text{ and } \tau_{n+1}(a) = -\frac{1+q}{\sqrt{q}}\tau_n(a).$$

3. The tower of affine Temperley-Lieb algebras and affine Markov trace

In this section we define a tower of affine Temperley-Lieb algebras, we show that this tower "surjects" onto the tower of Temperley-Lieb algebras mentioned in the introduction, and we define the affine Markov trace.

We consider the Dynkin diagram of the group $B(\tilde{A}_n)$. We denote the Dynkin automorphism $(\sigma_1 \mapsto \sigma_2 \mapsto ...\sigma_n \mapsto a_{n+1} \mapsto \sigma_1)$ by ψ_{n+1}. Notice that $\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}$ acts on $B(\tilde{A_{n-1}})$ as ψ_n as follows $(\sigma_1 \mapsto a_n \mapsto \sigma_{n-1} \mapsto \sigma_{n-2} \mapsto ..\sigma_2 \mapsto \sigma_1)$. We write $(\sigma_n..\sigma_1 a_{n+1})^d h = \psi^d\,[h]\,(\sigma_n..\sigma_1 a_{n+1})^d$, for any h in $B(\tilde{A_{n-1}})$, we keep same convention for the affine Temperley-Lieb algebra.



We have the following injection

$$\begin{aligned} G_n : K[B(\tilde{A_{n-1}})] &\longrightarrow K[B(\tilde{A}_n)] \\ \sigma_i &\longmapsto \sigma_i \qquad \text{for } 1 \le i \le n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1} \end{aligned}$$

We prove in [5], to which we refer for details, the following two propositions:

Proposition 3.1. *The injection G_n induces the following morphism of algebras:*

$$\begin{aligned} F_n : \widehat{TL}_n(q) &\longrightarrow \widehat{TL}_{n+1}(q) \\ t_{\sigma_i} &\longmapsto g_{\sigma_i} \textit{ for } 1 \le i \le n-1 \\ t_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}. \end{aligned}$$

Proposition 3.2. *The following map is a surjection of algebras*

$$\begin{aligned} E_n : \widehat{TL}_{n+1}(q) &\longrightarrow TL_n(q) \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \textit{ for } 1 \le i \le n \\ g_{a_{n+1}} &\longmapsto g_{\sigma_1} ... g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-1}}^{-1} ... g_{\sigma_1}^{-1}. \end{aligned}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd} \widehat{TL}_n(q) \arrow[r, "F_n"] \arrow[d, hook, two heads, "E_{n-1}"'] & \widehat{TL}_{n+1}(q) \arrow[d, two heads, "E_n"] \\ TL_{n-1}(q) \arrow[r, hook] & TL_n(q) \end{tikzcd}$$

Moreover, it is immediate that E_n composed with the natural inclusion of $TL_n(q)$ into $\widehat{TL}_{n+1}(q)$, gives $Id_{TL_n(q)}$.

In view of proposition 3.1 we can consider the tower of affine T-L algebras (it is not known whether it is a tower of faithful arrows or not):

$$\widehat{TL}_1(q) \xrightarrow{F_1} \widehat{TL}_2(q) \xrightarrow{F_2} \widehat{TL}_3(q) \longrightarrow \quad ... \quad \widehat{TL}_n(q) \xrightarrow{F_n} \widehat{TL}_{n+1}(q) \longrightarrow ...$$

Definition 3.3. *We call $(\hat{\tau}_n)_{1\leq n}$ an affine Markov trace, if every $\hat{\tau}_n$ is a trace function on $\widehat{TL}_n(q)$ with the following conditions:*

- $\hat{\tau}_1(1) = 1$, *(here $\widehat{TL}_1(q) = K$).*
- $\hat{\tau}_{n+1}(F_n(h)T_{\sigma_n}^{\pm 1}) = \hat{\tau}_n(h)$, *for all $h \in \widehat{TL}_n(q)$ and for $n \geq 1$.*
- $\hat{\tau}_n$ *is invariant under the Dynkin automorphism ψ_n for all n.*

Remark 3.4. *We notice that the second condition gives us that $\hat{\tau}_{n+1}\Big(F_n(h)T_{\sigma_n}^{-1}\Big) = \hat{\tau}_n\Big(h\Big)$, which means that:*

$$\hat{\tau}_{n+1}\Big(F_n(h)[\frac{1}{q^2}T_{\sigma_n} - \frac{q-1}{q\sqrt{q}}]\Big) = \hat{\tau}_n\Big(h\Big). \text{ Thus } \hat{\tau}_{n+1}\Big(F_n(h)\Big) = -\frac{q+1}{\sqrt{q}}\hat{\tau}_n\Big(h\Big).$$

Remark 3.5. *The third condition of definition 3.3 is, in fact, not independent, i.e., it results from the first and second conditions (see [3]). Nevertheless, we will keep viewing it as a condition.*

Remark 3.6. *This affine Markov trace does the job topologically, i.e., it gives an invariant for "affine oriented knots" and generalizes, in fact, the Jones invariant, noticing that the set of oriented knots in S^3 injects naturally into the set of "affine oriented knots". For further details see [5].*

Now, consider the following commutative diagram:

$$\begin{tikzcd}
\widehat{TL}_1(q) \arrow[r] \arrow[d, twoheadrightarrow] & \widehat{TL}_2(q) \arrow[r] \arrow[d, twoheadrightarrow] & \dots \arrow[r] & \widehat{TL}_n(q) \arrow[r] \arrow[d, twoheadrightarrow] & \widehat{TL}_{n+1}(q) \arrow[d, twoheadrightarrow] \\
TL_0(q) \arrow[r, hook] \arrow[drr, "\tau_1"'] & TL_1(q) \arrow[r, hook] \arrow[dr, "\tau_2"] & \dots \arrow[r, hook] & TL_{n-1}(q) \arrow[r, hook] \arrow[dl, "\tau_n"'] & TL_n(q) \arrow[dll, "\tau_{n+1}"] \\
 & & K & &
\end{tikzcd}$$

Set ρ_{n+1} to be the trace over $\widehat{TL}_{n+1}(q)$ induced by τ_{n+1} over $TL_n(q)$ for $0 \leq n$. We prove in [3] that $(\rho_i)_{1\leq i}$ is an affine Markov trace over $\left(\widehat{TL}_i(q)\right)_{1\leq i}$ and we prove the following Theorem:

Theorem 3.7. *[3]*
There exists a unique affine Markov trace over the tower of $\tilde{A}$-type Temperley-Lieb algebras, namely $(\rho_i)_{1\leq i}$.

The proof relies on Theorem 4.2 below, the proof of which separates into two cases: $n = 2$ and $n \geq 3$. The latter case is included in [3] while the former appears in the present note.

4. Markov elements and traces on $\widehat{TL}_{n+1}(q)$

4.1. **Markov elements.** We consider $F_n : \widehat{TL}_n(q) \longrightarrow \widehat{TL}_{n+1}(q)$ of proposition 3.1. In this subsection we set $F := F_n$. We give a definition of Markov elements in $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then we show that any trace over $\widehat{TL}_{n+1}(q)$ is uniquely determined by its values on those elements.

Definition 4.1. *For F as above, and $n \geq 2$, a Markov element in $\widehat{TL}_{n+1}(q)$ is any element of the form $Ag_{\sigma_n}^{\epsilon}B$, where A and B are in $F(\widehat{TL}_n(q))$ and $\epsilon \in \{0, 1\}$.*

The aim of this subsection is to prove the following theorem for $n = 2$.

Theorem 4.2. *[3] Let τ_{n+1} be any trace over $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then, τ_{n+1} is uniquely defined by its values on the Markov elements in $\widehat{TL}_{n+1}(q)$.*

The proof of theorem 4.2 for $n = 2$ is divided into two parts. In the first we show some general facts, in the second we give the proof for $n = 2$.

Part 1

In this part, we suppose that τ_{n+1} is any trace on $\widehat{TL}_{n+1}(q)$. We will apply τ_{n+1} to $\widehat{TL}_{n+1}(q)$ assuming that $2 \leq n$, and show that τ_{n+1} is uniquely determined on $\widehat{TL}_{n+1}(q)$ by its values on the positive powers of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, in addition to its values on Markov elements. From now on we denote by w: an arbitrary element in $W^c(\tilde{A}_n)$.

Lemma 4.3. *In $\widehat{TL}_{n+1}(q)$ we have:*

$$(1)\ g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k + \sum_{i=1}^{i=k-1} f_i (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i$$
$$+ A\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k g_{\sigma_n} \prod_{j=0}^{j=k-1} \psi^j \Big[F((t_{a_n})^{-1})\Big],$$
$$(2)\ (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n} = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k + \sum_{i=1}^{i=k-1} h_i (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i$$
$$+ A \prod_{j=0}^{j=k-1} \phi^j \Big[(g_{\sigma_{n-1}})^{-1}\Big] g_{\sigma_n} \Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k,$$

with A in the ground field, f_i, h_i in $F(\widehat{TL}_n(q))$ and $\phi^{-1} = \psi$.

Proof.

$$g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k = \Big(q-1\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k$$
$$+ q g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F\Big(t_{a_n}\Big) g_{\sigma_n} F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1}$$
$$= \Big(q-1\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k +$$
$$q g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F\Big(t_{a_n}\Big) g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1} \psi^{k-1}\Big[F((t_{a_n})^{-1})\Big].$$

So, by induction on k, (1) follows. In the very same way we deal with (2), by noticing that: $g_{a_{n+1}} g_{\sigma_n} = g_{\sigma_n}^{-1} F(t_{a_n}) g_{\sigma_n}^2 = (q-1) g_{a_{n+1}} + q g_{\sigma_n}^{-1} F(t_{a_n})$. □

A main result in [4] is to give a general form for "fully commutative braids", from which we deduce that any element of the basis of $\widehat{TL}_{n+1}(q)$ (where we have the convention $\sigma_{n+1} = 1$ in $W(\tilde{A}_n)$ thus $g_{\sigma_n\sigma_{n-1}..\sigma_i} = 1$ when $i = n+1$), is either of the form

$$c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}$$

or of the form

$$g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k d g_{\sigma_n\sigma_{n-1}..\sigma_i}$$

where c and d are in $F(\widehat{TL}_n(q))$, $1 \leq i \leq n+1$ and $0 \leq i_0 \leq n-1$.

By lemma 4.3 $c(g_{\sigma_n \sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n \sigma_{n-1}..\sigma_i}$ is of the form:

$$\sum_{j=1}^{j=h} c_j (g_{\sigma_n \sigma_{n-1}..\sigma_1 a_{n+1}})^j + M.$$

Where $h \leq k$, c_j is in $F(\widehat{TL}_n(q))$ for any j and M is a Markov element.

Now we deal with the second form:

$$\tau_{n+1}\Big(g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}\Big) = \tau_{n+1}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_i} g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big).$$

For any possible value for i_0 or i, we see that:

$$g_{\sigma_n\sigma_{n-1}..\sigma_i} g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = c' g_{\sigma_n} (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^s c'',$$

where c', c'' are in $F(\widehat{TL}_n(q))$ and $s \leq k+1$. By lemma 4.3 we see that this element is of the form:

$$\sum_{j=1}^{j=h} f_j (g_{\sigma_n \sigma_{n-1}..\sigma_1 a_{n+1}})^j + M,$$

where $h \leq k+1$, f_j is in $F(\widehat{TL}_n(q))$ for any j and M is a Markov element .

Hence, we see that in order to define τ_{n+1} uniquely it is enough to have its values on Markov elements and its values on $\Omega(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, where $1 \leq k$ (since if k is equal to 0 then we are again in the case of a Markov element) and Ω is in $F\Big(\widehat{TL}_n(q)\Big)$.

Lemma 4.4. *Let $2 \leq n$ then τ_{n+1} is uniquely defined by its values on Markov elements, in addition to its values on $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, with $0 \leq k$.*

Proof. In order to determine $\tau_{n+1}\Big(h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$, with a positive k and an arbitrary h in $F\Big(\widehat{TL}_n(q)\Big)$, it is enough to treat $\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$, with x in $W^c(\tilde{A_{n-1}})$, but the fact that τ_{n+1} is a trace, in addition to the fact that $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ acts as a Dynkin automorphism on $F\Big(\widehat{TL}_n(q)\Big)$, authorizes us to suppose that x has a reduced expression which ends with σ_{n-1}.

Now we show by induction on $l(x)$, that $\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$ is a sum of values of τ_{n+1} over $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, elements of the form $h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i$ with $i < k$ and Markov elements, (of course with coefficients in the ground ring which might be zeros).

For $l(x)=0$ the property is true. Take $l(x)>0$, and let $x=z\sigma_{n-1}$ be a reduced expression, hence:

$$\begin{aligned}\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k\Big) &= \tau_{n+1}\Big(F(t_z)F(t_{\sigma_{n-1}})g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\\ &= \tau_{n+1}\Big(F(t_z)\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}_{=-V(g_{\sigma_{n-1}},g_{\sigma_n})}g_{\sigma_{n-2}..\sigma_1a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big).\end{aligned}$$

Recalling that $V(g_{\sigma_{n-1}},g_{\sigma_n})=0$, this is equal to the following sum:

$$\begin{aligned}&-\tau_{n+1}\Big(F(t_z)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}g_{a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-2}..\sigma_1a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}\underbrace{g_{\sigma_n}g_{a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-2}..\sigma_1}\underbrace{g_{\sigma_n}g_{a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big).\end{aligned}$$

Now we apply the induction hypothesis to the first term. The second and the third terms are equal to:

$$\begin{aligned}&\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}\Big)^{k-1}\right)\\ &\quad+\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}\Big)^{k-1}\right),\end{aligned}$$

which is equal to:

$$\begin{aligned}&\tau_{n+1}\left(\psi^{1-k}\Big[F\Big((t_{a_n})^{-1}\Big)\Big]F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\right)\\ &\quad+\tau_{n+1}\left(\psi^{1-k}\Big[F\Big((t_{a_n})^{-1}\Big)\Big]F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\right).\end{aligned}$$

The fourth and the fifth terms are equal to:

$$\begin{aligned}&\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\right)\\ &\quad+\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1}\Big)\right).\end{aligned}$$

Thus, lemma 4.3 tells us that the property is true for those four terms. This step is to be applied repeatedly, to the powers of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ down to an element of the form $\tau_{n+1}\Big(h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^1\Big)$, arriving to the sum of:

$$\tau_{n+1}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})$$

and

$$\tau_{n+1}(h' g_{\sigma_{n-1}..\sigma_1 a_{n+1}}),$$

which is the sum of values of τ_{n+1} on Markov elements, since $h, h' \in F\Big(\widehat{TL}_n(q)\Big)$. $\square$

We end this part by the following lemma:

Lemma 4.5. *Let $1 \leq k$. Then $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$ is a sum of two kinds of elements:*

(1) $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^j g_{\sigma_n}h$, *with $j \leq k$.*
(2) $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^i g_{\sigma_n}f$, *with $i < k$,*

with h, f in $F\Big(\widehat{TL}_n(q)\Big)$ and $2 \leq n$.

Moreover, in the first type we have one, and only one element, with $j = k$, in which we have:

$$h = \prod_{i=0}^{i=k-1} \phi^i\Big[F(t_{a_n}^{-1})\Big].$$

Proof. Suppose that $k = 1$. Then,

$$g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}} = g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)g_{\sigma_n}F\Big(t_{a_n}\Big)^{-1},$$

the property is true.

Suppose the property is true for $k-1$, then, with $2 \leq k$, we have:

$$(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1} g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}.$$

We apply the property to $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}$, which gives two cases:

(1) $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'} g_{\sigma_n}h g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, with $j' \leq k-1$ which is: $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'} g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\psi^{-1}\Big[h\Big]$, which is equal to:

$$q g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'+1} g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[h\Big]$$

$$+ (q-1)g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\psi^{-1}\left[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\right]\psi^{-1}\left[h\right].$$

Since, $j'+1 \leq k$, the first term is clear to be of the first type, while the second term is equal to:

$$\Big(q-1\Big)qg_{\sigma_{n-1}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\Big]\psi^{-1}\Big[h\Big]+$$

$$\Big(q-1\Big)^2 g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\psi^{-1}\Big[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\Big]\psi^{-1}\Big[h\Big].$$

Here, the first term is of the second type (with $i=1<k$), and the second term is of the first type (with $j=1$).

(2) $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'}g_{\sigma_n}fg_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, with $i'<k-1$, which is:

$$\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'}g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\psi^{-1}\Big[f\Big]=$$

$$q\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'+1}g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[f\Big]+$$

$$\Big(q-1\Big)g_{\sigma_n}\Big(g_{\sigma_{n-1}..\sigma_1}F(t_{a_n})\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'}\Big]\psi^{-1}\Big[f\Big].$$

Since $i'+1<k$, the first term is of the second type, while the second term is of the first type with $j=1$. The lemma is proven.

(By induction over k again, the last formula is easy).

□

Part 2

In this part we will consider a given trace τ_3 over $\widehat{TL}_3(q)$. The aim is to show that τ_3 is uniquely defined by its values on Markov elements. consider

$$\begin{aligned} F_2 : \widehat{TL}_2(q) &\longrightarrow \widehat{TL}_3(q) \\ t_{\sigma_1} &\longmapsto g_{\sigma_1} \\ t_{a_n} &\longmapsto g_{\sigma_2}g_{a_3}g_{\sigma_2}^{-1}. \end{aligned}$$

In this part we will denote F_2 by F.

Lemma 4.4 tells that we can uniquely determine τ_3 by its values over $(g_{\sigma_2\sigma_1 a_3})^k$ for a positive k beside its values on Markov elements. We know as well by lemma 4.5 that $(g_{\sigma_2\sigma_1 a_3})^k$ is a sum of two kinds of elements:

(1) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^j g_{\sigma_2}h$ with $j \leq k$.

(2) $\Big(g_{\sigma_1}F(t_{a_2})\Big)^i g_{\sigma_2}f$ with $i<k$.

Here, h and f are in $F\Big(\widehat{TL}_2(q)\Big)$.

Moreover, in first type, only when $j=k$, we have:

$$h = \prod_{i=0}^{i=k-1} \psi^i\Big[\Big(F(t_{a_2})^{-1}\Big)\Big].$$

In other terms:

$$\begin{aligned} g_{\sigma_2}\Big(g_{\sigma_1}F\Big(t_{a_2}\Big)\Big)^k g_{\sigma_2} = {} & \Big(g_{\sigma_2\sigma_1a_3}\Big)^k \prod_{i=0}^{i=k-1} \psi^i\Big[\Big(F(t_{a_2})\Big)\Big] \\ & - \sum_{r=1}^{r=k-1} \Big(g_{\sigma_1}F\Big(t_{a_2}\Big)\Big)^r g_{\sigma_2} f_r \prod_{i=0}^{i=k-1} \psi^i\Big[\Big(F(t_{a_2})\Big)\Big] \\ & + \sum_{l=1}^{l=k-1} g_{\sigma_2}\Big(g_{\sigma_1}F\Big(t_{a_2}\Big)\Big)^l g_{\sigma_2} f'_l \prod_{i=0}^{i=k-1} \psi^i\Big[\Big(F(t_{a_2})\Big)\Big]. \end{aligned}$$

We repeat the same step on $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^l g_{\sigma_2}$ for every l. We deduce the following:

Corollary 4.6. *For every $h>0$, we have:* $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_2} = \sum_{j=0}^{j=h} c_j\Big(g_{\sigma_2\sigma_1a_3}\Big)^j + \sum_i M_i$.

Here, c_j is in $F\Big(\widehat{TL}_2(q)\Big)$ for every j, and M_i is a Markov element for every i.

Our way to prove Theorem 4.2 for $n=3$, is to show that $\tau_3\Big((g_{\sigma_2\sigma_1a_3})^k\Big)$ is a linear combination of values of τ_3 on Markov elements and values on elements of the form $c(g_{\sigma_2\sigma_1a_3})^h$, where $h<k$ and c in $F\Big(\widehat{TL}_2(q)\Big)$. Then, using the induction in the proof of Lemma 4.4, beside the fact that $\tau_3(g_{\sigma_2\sigma_1a_3})$ is a linear combination of some values of τ_3 on Markov elements, we see that the work is done.

Lemma 4.7. *Suppose that r and s are positives ,such that $r \le s$. Then:*

$$\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}g_{\sigma_2}\Big) = \sum_{j=0}^{j=h} c_j\Big(g_{\sigma_2\sigma_1a_3}\Big)^j + \sum_i M_i,$$

where $h \le s$, c_j is in $F\Big(\widehat{TL}_2(q)\Big)$ for every j and M_i is a Markov element for every i.

Proof.

$$\tau_3\bigg(\Big(g_{\sigma_1}\ F\ (t_{a_2})\Big)^r g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}g_{\sigma_2}\bigg)=$$

$$=\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}\underbrace{g_{\sigma_2\sigma_1a_3}\Big(g_{\sigma_2\sigma_1a_3}\Big)^{-1}}g_{\sigma_2}\bigg)$$

$$=\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_1}g_{\sigma_2}g_{\sigma_2\sigma_1a_3}\psi\Big[\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}\Big]\Big(g_{\sigma_2\sigma_1a_3}\Big)^{-1}g_{\sigma_2}\bigg)$$

$$=\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_1}g_{\sigma_2}g_{\sigma_2\sigma_1a_3}\psi\Big[\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}\Big]g_{a_3}^{-1}g_{\sigma_1}^{-1}\bigg)$$

$$=\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s F\Big(t_{a_2}\Big)g_{a_3}^{-1}F\Big(t_{a_2}\Big)\bigg)$$

$$=\frac{1-q}{q}\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s F\Big(t_{a_2}\Big)F\Big(t_{a_2}\Big)\bigg)$$

$$+\frac{1}{q}\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s F\Big(t_{a_2}\Big)g_{a_3}F\Big(t_{a_2}\Big)\bigg).$$

Now, the term corresponding to $\frac{1-q}{q}$ is τ_3 evaluated on the sum of Markov element and an element of style $c_j(g_{\sigma_2\sigma_1a_3})^1$. So, We are reduced to the second term, thus, reduced to:

$$\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s F\Big(t_{a_2}\Big)g_{a_3}F\Big(t_{a_2}\Big)\bigg).$$

Obviously, we are in the case:

$$q\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1} g_{\sigma_1}g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s F\Big(t_{a_2}\Big)g_{a_3}F\Big(t_{a_2}\Big)\bigg)=$$

$$q\tau_3\bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1} g_{\sigma_1}^2g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s F\Big(t_{a_2}^2\Big)g_{\sigma_2}\bigg),$$

since $g_{a_3}F\Big(t_{a_2}\Big)=F\Big(t_{a_2}\Big)g_{\sigma_2}$.

Now,

$$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{r-1}g_{\sigma_1}^2 g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s F\Big(t_{a_2}^2\Big)g_{\sigma_2}\Big) =$$

$$\Big(q-1\Big)\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1}g_{\sigma_1}^2 g_{a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_2}\Big)$$

$$+\ q\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1}g_{\sigma_1}^2 g_{a_3}F(t_{a_2})g_{\sigma_1}\Big(F(t_{a_2})g_{\sigma_1}\Big)^{s-1}g_{\sigma_2}\Big).$$

Which is equal to the sum:

$$\Big(q-1\Big)\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1}g_{\sigma_1}^2 F\Big(t_{a_2}\Big)\underbrace{g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_2}}\Big)$$

$$+q\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1}g_{\sigma_1}^2 F\Big(t_{a_2}\Big)g_{\sigma_2}g_{\sigma_1}\Big(F(t_{a_2})g_{\sigma_1}\Big)^{s-1}g_{\sigma_2}\Big).$$

Now, the first term is covered by corollary 4.6. Thus we are interested with the second term:

$$\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1}g_{\sigma_1}^2\ F\ \Big(t_{a_2}\Big)g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}\Big) =$$

$$q\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}\Big)$$

$$+\Big(q-1\Big)\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-2}g_{\sigma_1}F(t_{a_2}^2)g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}\Big).$$

Which is equal to:

$$q\tau_3\Big(\underbrace{g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_2}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}\Big)+$$

$$\Big(q-1\Big)\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-2}g_{\sigma_1}F\Big(t_{a_2}^2\Big)g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}\Big)$$

The first term is covered by corollary 4.6. We are reduced to

$$\Big(\tau_3\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-2}g_{\sigma_1}F\Big(t_{a_2}^2\Big)g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}\Big),$$

which is equal to:

$$\Big(q-1\Big)\tau_3\Bigg(\underbrace{g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1}g_{\sigma_2}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}\Bigg)+$$

$$q\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-2}g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}\Bigg).$$

The first term is covered by corollary 4.6. Thus, we see that, in general, the value of τ_3 over $\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r}g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s}g_{\sigma_1}g_{\sigma_2}$ can be shifted to its value over:

$$\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-2}g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}.$$

After a finite number of repetitions of the computation above (with the possibility of exchanging r and s), we see that the lemma is proven modulo determining:

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{m}\underbrace{g_{\sigma_1}g_{\sigma_2}g_{\sigma_1}}F\Big(t_{a_2}\Big)g_{\sigma_1}g_{\sigma_2}\Bigg).$$

We see that the terms corresponding to $-g_{\sigma_1}$ and -1 correspond to Markov elements. While those who correspond to $-g_{\sigma_1}g_{\sigma_2}$ and $-g_{\sigma_2}$ are covered by corollary 4.6 for $h=1$. Finally the term corresponding to $-g_{\sigma_2}g_{\sigma_1}$ is covered by corollary 4.6 for $h=m$.

□

Lemma 4.8. *Suppose that r and s are positive such that $r \leq s$. Then:*

$$\tau_3\Bigg(g_{a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s}g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r}\Bigg)=\sum_{j=0}^{j=h}c_j\Big(g_{\sigma_2\sigma_1a_3}\Big)^{j}+\sum_{i}M_i.$$

Where $h \leq s$, c_j is in $F(\widehat{TL}_2(q))$ for every j and M_i is a Markov element for every i.

Proof.

$$\tau_3\Bigg(g_{a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s}g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r}\Bigg)=$$

$$\tau_3\Bigg(g_{a_3}\Big(g_{\sigma_2\sigma_1a_3}\Big)^{-1}g_{\sigma_2\sigma_1a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s}g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r}\Bigg)=$$

$$\tau_3\Bigg(g_{a_3}\Big(g_{\sigma_2\sigma_1a_3}\Big)^{-1}\psi\Big[F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s}\Big]g_{\sigma_2\sigma_1a_3}g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r}\Bigg)=$$

$$\tau_3\Bigg(g_{\sigma_1}^{-1}g_{\sigma_2}^{-1}g_{\sigma_1}\Big(F(t_{a_2})g_{\sigma_1}\Big)^{s}g_{\sigma_2}g_{\sigma_1}g_{a_3}^{2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r}\Bigg).$$

Here, we see that this term is a sum of two terms coming from $g_{a_3}^2=(q-1)g_{a_3}+q$. The term corresponding to $(q-1)g_{a_3}$ is covered the same way as in the last lemma (with a_3

instead of σ_2 above. Hence we treat the term corresponding to q, that is:

$$\tau_3\left(g_{\sigma_1}^{-1}\underbrace{g_{\sigma_2}^{-1}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s\underbrace{g_{\sigma_1}g_{\sigma_2}g_{\sigma_1}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\right).$$

Before applying TL relations, we see in the same way as above, that we are reduced to the next value (otherwise, it is τ_3 evaluated on a Markov element):

$$\tau_3\left(g_{\sigma_1}^{-1}\underbrace{g_{\sigma_2}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s\underbrace{g_{\sigma_1}g_{\sigma_2}g_{\sigma_1}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\right).$$

We see that the terms corresponding to $-g_{\sigma_1}$ and -1 correspond to Markov elements. And those who correspond to $-g_{\sigma_2 g_{\sigma_1}}$ and $-g_{\sigma_2}$, are covered by corollary 4.6 for $h = s$.

The term corresponding to $-g_{\sigma_1}g_{\sigma_2}$ is:

$$\tau_3\left(g_{\sigma_1}^{-1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\right),$$

which is:

$$\frac{1-q}{q}\tau_3\left(\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}\underbrace{g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_2}}\right)+\frac{1}{q}\tau_3\left(g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\right)$$

The first term is covered by corollary 4.6 for $h = r$. The second follows by lemma 4.7.

□

Let us go back to $\tau_3(g_{\sigma_2\sigma_1 a_{n+1}})^k$. The aim is to show that:

$$\tau_3\Big(g_{\sigma_2\sigma_1 a_{n+1}}\Big)^k = \tau_3\left(\sum_{j=0}^{j=h} c_j (g_{\sigma_2\sigma_1 a_3})^j + \Sigma_i M_i\right),$$

where $h < k$. By lemma 4.5, it is sufficient to deal with:

$$g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\prod_{i=0}^{i=k-1}\psi^i\Big[\Big(F(t_{a_2})^{-1}\Big)\Big].$$

It is clear that this element is written as a linear combination of four kind of elements:

1) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h$.

2) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}$.

3) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h F\Big(t_{a_2}\Big)$.

4) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h$,

where $h \leq [\frac{k}{2}] < k$, since $1 < k$.

1) We start by $\tau_3\Big(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h\Big)$. Which is equal to:

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k \underbrace{g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_2}}\Bigg),$$

follows directly, regarding corollary 4.6 .

2) Now we consider

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}\Bigg), \tag{1}$$

which is equal to:

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}g_{\sigma_2\sigma_1 a_{n+1}}\Big(g_{\sigma_2\sigma_1 a_3}\Big)^{-1}\Bigg) =$$

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}^2 g_{\sigma_1}g_{a_3}\psi\Big[\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}\Big]\Big(g_{\sigma_2\sigma_1 a_3}\Big)^{-1}\Bigg) =$$

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_2}^2 g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h F(t_{a_2})g_{a_3}^{-1}F\Big(t_{a_2}\Big)\Bigg),$$

with the very same steps as used above, we see that we are reduced to :

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2}))^{k-1} g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h F\Big(t_{a_2}\Big)g_{a_3}F\Big(t_{a_2}\Big)\Bigg), \text{ which is:}$$

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}\underbrace{g_{a_3}F\Big(t_{a_2}\Big)}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h \underbrace{g_{a_3}F\Big(t_{a_2}\Big)}\Bigg) =$$

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}\underbrace{F\Big(t_{a_2}\Big)g_{\sigma_2}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h \underbrace{F\Big(t_{a_2}\Big)g_{\sigma_2}}\Bigg), \text{ which is equal to:}$$

$$(q-1)\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_2}\Bigg) + q\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1} g_{\sigma_1}g_{\sigma_2}\Bigg),$$

we see that corollary 4.6 covers the first term. Thus we see that:

$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}\Bigg)$ in (eq. 1), is shifted to:

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1} g_{\sigma_1}\Bigg),$$

going on in this manner, we arrive to:

$$\tau_3\Big(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}\Big)g_{\sigma_1}\Big),$$

with the same steps above, we see that we are reduced to

$$\tau_3\Big(\Big(g_{\sigma_1}F\Big(t_{a_2}\Big)\Big)^k g_{\sigma_2}g_{\sigma_1}F(t_{a_2}^2)g_{\sigma_2}\Big), \text{ which is equal to}$$

$$\Big(q-1\Big)\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}\Big)g_{\sigma_2}\Big)+q\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}g_{\sigma_2}\Big),$$

corollary 4.6 and TL relations end the job.

3) Here we deal with $\tau_3\Big(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h F\Big(t_{a_2}\Big)\Big)$, which is:

$$\begin{aligned}
&\tau_3\Big(F\Big(t_{a_2}\ \Big)\ g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h\Big)\\
&\qquad= \tau_3\Big(g_{a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h\Big)\\
&\qquad= \tau_3\Big(g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^k F\Big(t_{a_2}\Big)g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h\Big),
\end{aligned}$$

but, $g_{\sigma_2}=F\Big(t_{a_2}^{-1}\Big)g_{a_3}F\Big(t_{a_2}\Big)$, thus:

$$\begin{aligned}
&\tau_3\Big(g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^k g_{a_3}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h\Big)\\
&= \tau_3\Big(g_{a_3}g_{\sigma_2\sigma_1a_3}^{-1}g_{\sigma_2\sigma_1a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^k g_{a_3}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h\Big)\\
&= \tau_3\Big(g_{\sigma_2}^{-1}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2\sigma_1a_3}g_{a_3}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h g_{\sigma_1}^{-1}\Big),
\end{aligned}$$

as we have done above, using the quadratic relations, we see that we are reduced to:

$$\tau_3\Big(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h g_{\sigma_1}^{-1}\Big)$$

$$= \tau_3\Big(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}^2\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\Big)$$

$$= \Big(q-1\Big)\tau_3\Big(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h\Big)+$$

$$q\tau_3\Big(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}g_{\sigma_1}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\Big),$$

the first term is covered by corollary 4.6. For the second we see that it is equal to:

$$q\tau_3\Big(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\Big),\ \text{which is equal to}$$

$$q\Big(q-1\Big)\tau_3\Big(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\Big)+$$

$$q^2\tau_3\Big(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-2}\Big),$$

the first term is obviously, covered by corollary 4.6, for the second one we see that it is case 3 itself, but with $h-2$ instead of h. Thus, we get two elements for τ_3 to be evaluated on:

[a] $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}F\Big(t_{a_2}\Big)g_{\sigma_1}F\Big(t_{a_2}\Big)$,

[b] $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^2$.

For [b] we can repeat what we have done until arriving to:

$$\tau_3\Big(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}g_{\sigma_1}F\Big(t_{a_2}\Big)\Big),\ \text{which is the following sum:}$$

$$\Big(q-1\Big)\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}\Big)g_{\sigma_2}\Big)+q\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k\underbrace{g_{\sigma_2}F\Big(t_{a_2}\Big)g_{\sigma_2}}_{F(t_{a_2})g_{\sigma_2}F(t_{a_2})}\Big),$$

obviously, the first term is covered by corollary 4.6, the second term is a Markov element.

For [a] we see that:

$$\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^k g_{\sigma_2}F\big(t_{a_2}\big)g_{\sigma_1}F\big(t_{a_2}\big)\Big)$$

$$=\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}\underbrace{F\big(t_{a_2}\big)g_{\sigma_2}F\big(t_{a_2}\big)}_{g_{\sigma_2}^2 g_{a_3}}g_{\sigma_1}F\big(t_{a_2}\big)\Big)$$

$$= \quad \big(q-1\big)\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}g_{\sigma_2}g_{a_3}g_{\sigma_1}F\big(t_{a_2}\big)\Big)+$$

$$q\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}g_{a_3}g_{\sigma_1}F\big(t_{a_2}\big)\Big),$$

the first term is covered by corollary 4.6, since it is equal to:

$$\big(q-1\big)\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}F\big(t_{a_2}\big)g_{\sigma_2}g_{\sigma_1}F\big(t_{a_2}\big)g_{\sigma_2}\Big).$$

For the second term, we see that:

$$q\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}g_{a_3}g_{\sigma_1}F\big(t_{a_2}\big)\Big)$$

$$=q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}g_{a_3}g_{\sigma_1}F\big(t_{a_2}\big)g_{\sigma_2}\Big)$$

$$=q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}g_{a_3}g_{\sigma_1}g_{a_3}F\big(t_{a_2}\big)\Big)$$

$$=q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1} g_{\sigma_1}^2 g_{a_3}g_{\sigma_1}F\big(t_{a_2}\big)\Big),$$

which is a Markov element, since $g_{a_3}=F(t_{a_2})g_{\sigma_2}F(t_{a_2}^{-1})$.

4) We deal with $\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^k g_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^h\Big)$, using the same techniques:

$$\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^k g_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^h\Big)$$

$$=\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^k g_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^h g_{\sigma_2\sigma_1 a_3}g_{\sigma_2\sigma_1 a_3}^{-1}\Big)$$

$$=\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^k g_{\sigma_2}^2 g_{\sigma_1}g_{a_3}\big(g_{\sigma_1}F(t_{a_2})\big)^h g_{\sigma_2\sigma_1 a_3}^{-1}\Big),$$

so, we are reduced to:

$$\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-1}g_{\sigma_1}g_{a_3}\left(g_{\sigma_1}F(t_{a_2})\right)^{h}g_{a_3}F\left(t_{a_2}\right)\right).$$ Which is equal to:

$$\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-1}\underbrace{g_{\sigma_1}g_{a_3}g_{\sigma_1}}_{V(g_{\sigma_1},g_{a_3})}F\left(t_{a_2}\right)\left(g_{\sigma_1}F(t_{a_2})\right)^{h-1}F\left(t_{a_2}\right)g_{\sigma_2}\right),$$

for -1 and $-g_{\sigma_1}$ it is a Markov element. For $-g_{a_3}g_{\sigma_1}$ we see that:

$$\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-1}g_{a_3}g_{\sigma_1}F\left(t_{a_2}\right)\left(g_{\sigma_1}F(t_{a_2})\right)^{h-1}F\left(t_{a_2}\right)g_{\sigma_2}\right)$$
$$= \tau_3\left(g_{a_3}F\left(t_{a_2}\right)\left(g_{\sigma_1}F(t_{a_2})\right)^{k-1}g_{a_3}\left(g_{\sigma_1}F(t_{a_2})\right)^{h}\right),$$

which is covered by lemma 4.8.

For $-g_{a_3}$, we see that:

$$\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-1}g_{a_3}F\left(t_{a_2}\right)\left(g_{\sigma_1}F(t_{a_2})\right)^{h-1}F\left(t_{a_2}\right)g_{\sigma_2}\right)$$
$$= \tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-2}g_{\sigma_1}F\left(t_{a_2}^2\right)g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{h-2}g_{\sigma_1}F\left(t_{a_2}^2\right)g_{\sigma_2}\right)$$
$$= \left(q-1\right)\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-1}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{h-2}g_{\sigma_1}F\left(t_{a_2}^2\right)g_{\sigma_2}\right)+$$
$$q\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-2}g_{\sigma_1}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{h-2}g_{\sigma_1}F\left(t_{a_2}^2\right)g_{\sigma_2}\right),$$

the first term is covered by corollary 4.6. We do the same thing with $F(t_{a_2}^2)$ in the second term, we arrive to:

$$q^2\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-2}g_{\sigma_1}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{h-2}g_{\sigma_1}g_{\sigma_2}\right),$$

which is the case of lemma 4.7.

For $-g_{\sigma_1}g_{a_3}$ we see that:

$$\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k-1}g_{\sigma_1}g_{a_3}F\left(t_{a_2}\right)\left(g_{\sigma_1}F(t_{a_2})\right)^{h-1}F\left(t_{a_2}\right)g_{\sigma_2}\right)$$

$$= \tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{h-1}F\left(t_{a_2}\right)g_{\sigma_2}\right)$$

$$= \left(q-1\right)\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{h-1}g_{\sigma_2}\right)+$$

$$q\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{k}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{h-2}g_{\sigma_1}g_{\sigma_2}\right),$$

corollary 4.6 covers the first term, while the second term is covered by (1) from our four cases.

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